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Weighted Polynomial Approximation of Entire Functions, II

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Necessary and sufficient conditions are given for a function f defined almost everywhere on the real line to have an extension to the complex plane as an entire function of specified order and finite type. These conditions are in terms of the degree of approximation of f by polynomials in weighted L^p norms.

1. INTRODUCTION

In the first part of this paper [6], we gave a characterization of an entire function of finite exponential type in terms of the constructive properties of its restriction to the real line. The purpose of this part is to give a similar characterization of entire functions of finite order and type. These results are the counterparts in the theory of weighted polynomial approximation on the whole real line of the following theorems of Varga and Bernstein in the theory of polynomial approximation on compact intervals.

Let f be a real valued continuous function on $[-1, 1]$. Put, for every positive integer n,

$$E_n(f) = \inf \max_{-1 \leqslant x \leqslant 1} |f(x) - P(x)|, \tag{1.1}$$

where the inf is over all polynomials P of degree at most n.

THEOREM 1 (Varga [7]). *If*

$$A = \limsup_{n\to\infty} \left\{ \frac{n \log n}{-\log E_n(f)} \right\} < \infty, \tag{1.2}$$

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then f has an extension to the complex plane as an entire function of finite order Λ, i.e.,

$$\limsup_{R\to\infty} \left\{\frac{\log\log \max_{|z|=R} |f(z)|}{\log R}\right\} = \Lambda. \tag{1.3}$$

Conversely, if f is the restriction to $[-1, 1]$ of an entire function of order Λ, then (1.2) *holds.*

THEOREM 2 (Bernstein [1]). *If, for some $\Lambda > 0$,*

$$\limsup_{n\to\infty} \{n!^{1/\Lambda} E_n(f)\}^{1/n} < \infty, \tag{1.4}$$

then f has an extension to the complex plane as an entire function $f(z)$ of order Λ and of finite type, i.e.,

$$\limsup_{R\to\infty} \left\{\frac{\log \max_{|z|=R} |f(z)|}{R^{\Lambda}}\right\} < \infty. \tag{1.5}$$

Conversely, if f is the restriction to $[-1, 1]$ of an entire function of positive order Λ and of finite type, then (1.4) *holds.*

Here, as in the first part of this paper, we shall consider functions defined almost everywhere on the whole real line and give the necessary and sufficient conditions for such a function to have an entire extension of a specified order and finite type. These conditions will be stated in terms of the degrees of approximation of f by polynomials in weighted L^p norms.

2. MAIN RESULTS

We consider weights of the form $w_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha \geqslant 2$.
For a Lebesgue measurable function g on $\mathbb{R}$, put

$$\|g\|_p = \left(\int_{\mathbb{R}} |g(x)|^p \, dx\right)^{1/p}, \qquad 1 \leqslant p < \infty.$$

$$\|g\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |g(x)|.$$

If $w_\alpha f \in L^p(\mathbb{R})$ and n is a nonnegative integer, put

$$\varepsilon_n(p, \alpha, f) = \inf \|w_\alpha(f-P)\|_p, \tag{2.1}$$

where the inf is taken over all polynomials of degree at most $n-1$. We denote the class of all polynomials of degree at most n by π_n.

THEOREM 1. *Let $\alpha \geqslant 2$, $1 \leqslant p \leqslant \infty$ and $w_\alpha f \in L^p(\mathbb{R})$. Suppose*

$$\mu(p, \alpha, f) = \limsup_{n\to\infty} \frac{n \log n}{-\log \varepsilon_n(p, \alpha, f)} < \infty. \tag{2.2}$$

Then f has an extension to the complex plane as an entire function of order $\lambda < \alpha$ given by

$$\frac{1}{\lambda} - \frac{1}{\alpha} = \frac{1}{\mu(p, \alpha, f)}. \tag{2.3}$$

Conversely, if $\lambda < \alpha$ and f is the restriction to $\mathbb{R}$ of an entire function of finite order λ then for each $p \geqslant 1$, $w_\alpha f \in L^p(\mathbb{R})$, $\mu(p, \alpha, f)$ is finite and (2.3) holds.

THEOREM 2. *Let $\alpha \geqslant 2$, $1 \leqslant p \leqslant \infty$, $0 < \lambda < \alpha$ and $w_\alpha f \in L^p(\mathbb{R})$. Suppose*

$$\rho_1(p, \alpha, f) = \limsup_{n\to\infty} \{n!^{1/\lambda - 1/\alpha} \varepsilon_n(p, \alpha, f)\} < \infty. \tag{2.4}$$

Then f has an extension to the complex plane as an entire function of order λ and of finite type, say, τ. Further, there exist positive constants c_1 and c_2 depending upon α only such that

$$c_1 \rho_1(p, \alpha, f) \leqslant (\tau\lambda)^{1/\lambda} \leqslant c_2 \rho_1(p, \alpha, f). \tag{2.5}$$

Conversely, if f is the restriction to $\mathbb{R}$ of an entire function of order λ and finite type τ then, for each $p \geqslant 1$, $w_\alpha f \in L^p(\mathbb{R})$, $\rho_1(p, \alpha, f)$ is finite and (2.5) holds.

3. PROOFS

As in the first part of our paper [6], we shall prove the theorems first for $p = 2$ and then extend them to other values of p.

Let $\{p_k(\alpha, x) = p_k(x)\}_{k=0}^{\infty}$ be the family of orthonormal polynomials with respect to the weight w_α^2. For $w_\alpha f \in L^p(\mathbb{R})$, $p \geqslant 1$ we have the Fourier orthonormal expansion

$$f(x) \sim \sum b_k p_k(x), \tag{3.1}$$

where

$$b_k = \int_{\mathbb{R}} f(t)\, p_k(t)\, w_\alpha^2(t)\, dt. \tag{3.2}$$

We then have for $w_\alpha f \in L^2(\mathbb{R})$,

$$\varepsilon_n(2, \alpha, f) = \left(\sum_{k=n}^{\infty} b_k^2 \right)^{1/2}. \tag{3.3}$$

We shall denote $n^{1/\alpha}$ by q_n. The following proposition will be used heavily during the proof:

PROPOSITION 1 [3–5]. (a) *There exists a constant c_3 such that for every $P \in \pi_n$,*

$$\| w_\alpha P' \|_2 \leqslant c_3 (n/q_n) \| w_\alpha P \|_2 = c_3 n^{1-1/\alpha} \| w_\alpha P \|_2. \tag{3.4}$$

(b) *There exist constants c_4 and c_5 such that for each polynomial $P \in \pi_n$,*

$$c_4 (n^{-1/\alpha})^{1/p - 1/r} \| w_\alpha P \|_p \leqslant \| w_\alpha P \|_r \leqslant c_5 (n^{1-1/\alpha})^{1/p - 1/r} \| w_\alpha P \|_p, \tag{3.5}$$

where $1 \leqslant p < r \leqslant \infty$ and $c_4, c_5 > 0$ depend only on α, p and r.

(c) *There exists a constant c_6 such that*

$$\varepsilon_k(2, \alpha, x^n) \leqslant c_6 k^{1/\alpha - 1} \varepsilon_{k-1}(2, \alpha, n x^{n-1}) \tag{3.6}$$

for each n and for each $k \leqslant n$.

Finally, notice that

$$\mu(p, \alpha, f) = \limsup_{n \to \infty} \log n! / -\log \varepsilon_n(p, \alpha, f). \tag{3.7}$$

LEMMA 2. *Suppose $\mu(2, \alpha, f) = \mu$ in (2.2) be finite and $\infty > \beta > \mu$. Then there exist constants $c_7 = c_7(\alpha, \mu, f)$ and $c_8 = c_8(\alpha)$ such that for sufficiently large n,*

$$\sum_{k=n}^{\infty} |b_k| \, \| w_\alpha p_k^{(n)} \|_\infty \leqslant c_7 c_8^n n!^{1 - 1/\alpha - 1/\beta}. \tag{3.8}$$

Proof. Choose N such that $n \geqslant N$ implies

$$|b_k| \leqslant \varepsilon_k(2, \alpha, f) \leqslant \left(\frac{1}{k!} \right)^{1/\beta}; \qquad k \geqslant n. \tag{3.9}$$

From (3.9), (3.5) and repeated applications of (3.4), we get, for $n \geqslant N$,

$$\begin{aligned}
&\sum_{k=n}^{\infty} |b_k| \, \|w_\alpha p_k^{(n)}\|_\infty \\
&\quad \leqslant c_5 \sum_{k=n}^{\infty} |b_k| [k-n]^{(1/2)(1-1/\alpha)} \|w_\alpha p_k^{(n)}\|_2 \\
&\quad \leqslant c_5 c_3^n \sum_{k=n}^{\infty} |b_k| [k-n]^{(1/2)(1-1/\alpha)} \left[\frac{k!}{(k-n)!}\right]^{1-1/\alpha} \|w_\alpha p_k\|_2 \\
&\quad = c_5 c_3^n \sum_{k=n}^{\infty} |b_k| [k-n]^{(1/2)(1-1/\alpha)} \left[\frac{k!}{(k-n)!}\right]^{1-1/\alpha} \\
&\quad \leqslant c_5 c_3^n \sum_{k=n}^{\infty} \left(\frac{1}{k!}\right)^{1/\beta} [k-n]^{(1/2)(1-1/\alpha)} \left[\frac{k!}{(k-n)!}\right]^{1-1/\alpha} \\
&\quad = c_5 c_3^n \sum_{k=n}^{\infty} k!^{1-1/\alpha-1/\beta} [k-n]^{(1/2)(1-1/\alpha)} \left[\frac{1}{(k-n)!}\right]^{1-1/\alpha}
\end{aligned} \tag{3.10}$$

Case I. $1-1/\alpha-1/\beta \leqslant 0$. Then a straightforward estimation in (3.10) gives (3.8) with $c_8 = c_3$.

Case II. $1-1/\alpha-1/\beta > 0$. Then, by Hölder's inequality, the binomial theorem and the ratio test,

$$\begin{aligned}
&\sum_{k=n}^{\infty} k!^{1-1/\alpha-1/\beta} (k-n)^{(1/2)(1-1/\alpha)} \left[\frac{1}{(k-n)!}\right]^{1-1/\alpha} \\
&\quad = \sum_{k=n}^{\infty} \left[\frac{k!}{(k-n)!\, n!} \frac{1}{2^k}\right]^{1-1/\alpha-1/\beta} \\
&\qquad \times n!^{1-1/\alpha-1/\beta} 2^{k(1-1/\alpha-1/\beta)} \frac{(k-n)^{(1/2)(1-1/\alpha)}}{(k-n)!^{1/\beta}} \\
&\quad \leqslant n!^{1-1/\alpha-1/\beta} 2^{n(1-1/\alpha-1/\beta)} \left(\sum_{k=0}^{\infty} \binom{k+n}{k} \frac{1}{2^{k+n}}\right)^{1-1/\alpha-1/\beta} \\
&\qquad \times \left(\sum_{k=0}^{\infty} \frac{2^{k(1-1/\alpha-1/\beta)(\alpha\beta/(\alpha+\beta))} k^{(1/2)(1-1/\alpha)(\alpha\beta/(\alpha+\beta))}}{k!^{\alpha/(\alpha+\beta)}}\right)^{(\alpha+\beta)/\alpha\beta} \\
&\quad \leqslant c_9 n!^{1-1/\alpha-1/\beta} c_{10}^n.
\end{aligned}$$

This proves the lemma with $c_7 = c_5 c_9$ and $c_8 = c_3 c_{10}$ even in this case. ∎

In view of Lemma 2, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=n}^{\infty} |b_k|\, |p_k^{(n)}(0)|\, |z|^n$$

converges uniformly on compact subsets of the complex plane.

Interchanging the order of summation, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=n}^{\infty} b_k\, p_k^{(n)}(0)\, z^n &= \sum_{k=0}^{\infty} b_k \sum_{n=0}^{k} \frac{p_k^{(n)}(0)}{n!} z^n \\ &= \sum_{k=0}^{\infty} b_k\, p_k(z). \qquad (3.11)\end{aligned}$$

The last series thus converges uniformly on compact subsets of the complex plane to an entire function, say, $g(z)$. It follows that the restriction of g to the real line is almost everywhere equal to f. Further, for g, we have the power series

$$g(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{k=n}^{\infty} b_k\, p_k^{(n)}(0)\right) z^n = \sum_{n=0}^{\infty} a_n z^n. \qquad (3.12)$$

So,

$$|a_n| \leqslant \frac{1}{n!} \sum_{k=n}^{\infty} |b_k|\, |p_k^{(n)}(0)| \leqslant c_7 c_8^n (n!)^{-1/\alpha - 1/\beta}.$$

Hence,

$$\log \frac{1}{|a_n|} \geqslant \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \log n! - n \log c_8 - \log c_7.$$

It follows that g is of finite order λ given by [2, Theorem 2.2.2, p. 9]

$$\lambda = \limsup_{n\to\infty} \frac{\log n!}{\log 1/|a_n|}, \qquad (3.13)$$

where

$$\frac{1}{\lambda} \geqslant \frac{1}{\alpha} + \frac{1}{\beta} \qquad (\text{so that } \lambda < \alpha).$$

Since $\beta > \mu$ was arbitrary, this implies

$$\frac{1}{\lambda} \geqslant \frac{1}{\alpha} + \frac{1}{\mu}. \qquad (3.14)$$

It remains to show that the converse holds and that equality holds in (3.14). For this purpose, we recall an estimate from the first part of this paper [6, Proposition 3.3].

PROPOSITION 3. *There exist constants* c_{11} *and* c_{12} *depending upon* α *alone such that, for every nonnegative integer* r,

$$\| w_\alpha x^r \|_2 \leqslant c_{11} c_{12}^r (r!)^{1/\alpha}. \tag{3.15}$$

Now, let f be the restriction to the real line of an entire function having finite order $\lambda < \alpha$ and $\sum a_n z^n$ be its Taylor expansion. Then, clearly, $w_\alpha f \in L^2(\mathbb{R})$. Let $\lambda < \beta < \alpha$. Then, by (3.13), we can choose N so that $n \geqslant N$ implies

$$\frac{\log n!}{\log 1/|a_n|} < \beta,$$

and hence,

$$|a_n| < \left(\frac{1}{n!}\right)^{1/\beta}. \tag{3.16}$$

Now, we have, by Proposition 3, for all $k \geqslant N$,

$$\begin{aligned} \sum_{n=k+1}^{\infty} |a_n| \, \| w_\alpha x^n \|_2 &\leqslant c_{11} \sum_{n=k+1}^{\infty} \left(\frac{1}{n!}\right)^{1/\beta} c_{12}^n n^{1/\alpha} \\ &\leqslant c_{13} c_{12}^k \left(\frac{1}{k!}\right)^{1/\beta - 1/\alpha}. \end{aligned} \tag{3.17}$$

Hence,

$$\varepsilon_k(2, \alpha, f) \leqslant c_{13} c_{12}^k \left(\frac{1}{k!}\right)^{1/\beta - 1/\alpha};$$

$$-\log \varepsilon_k(2, \alpha, f) \geqslant \left(\frac{1}{\beta} - \frac{1}{\alpha}\right) \log k! - k \log c_{12} - \log c_{13}, \qquad k \geqslant N.$$

Hence, $\mu < \infty$ and

$$\frac{1}{\mu} \geqslant \frac{1}{\beta} - \frac{1}{\alpha}.$$

Recall that $\beta > \lambda$ was arbitrary, so that

$$\frac{1}{\lambda} \leqslant \frac{1}{\alpha} + \frac{1}{\mu}. \tag{3.18}$$

In view of (3.18) and (3.14), this completes the proof of Theorem 2.1 in the case when $p=2$. To extend the theorem for other values of p, let $\mu(p, \alpha, f) < \infty$ for some $p \geqslant 1$ and choose $\tau_n \in \pi_n$ so that

$$\|w_\alpha(f-\tau_n)\|_p \leqslant 2\varepsilon_n(p, \alpha, f), \qquad n \geqslant 0. \tag{3.19}$$

Putting $P_n = \tau_{n+1} - \tau_n$, we get,

$$f = \sum_{n=0}^{\infty} P_n + \tau_0 \tag{3.20}$$

in the sense that

$$\left\| w_\alpha \left(f - \tau_0 - \sum_{k=0}^{N} P_k\right)\right\|_p \to 0 \qquad \text{as} \quad N \to \infty. \tag{3.20$'$}$$

Also,

$$\|w_\alpha P_n\|_p \leqslant 4\varepsilon_n(p, \alpha, f). \tag{3.21}$$

Choose $\beta > \mu(p, \alpha, f)$, and K large enough so that $k \geqslant K$ implies

$$\varepsilon_k(p, \alpha, f) \leqslant \left(\frac{1}{k!}\right)^{1/\beta}. \tag{3.22}$$

Then, if $1 \leqslant p, r \leqslant \infty$ and $k \geqslant K$, (3.5), (3.21) and (3.22) imply that

$$\begin{aligned} \sum_{n=k}^{\infty} \|w_\alpha P_n\|_r &\leqslant c_{14} \sum_{n=k}^{\infty} n^{(1-1/\alpha)(1/p-1/r)} \|w_\alpha P_n\|_p \\ &\leqslant c_{15} \sum_{n=k}^{\infty} n^{(1-1/\alpha)} \left(\frac{1}{n!}\right)^{1/\beta}. \end{aligned} \tag{3.23}$$

Now, choose an integer s such that

$$1 - \frac{1}{\alpha} - \frac{s}{\beta} \leqslant 0.$$

Then,

$$\begin{aligned} \sum_{n=k}^{\infty} \|w_\alpha P_n\|_r \leqslant c_{15} \sum_{n=k}^{\infty} & \frac{n^{1-1/\alpha-s/\beta}}{[(1-1/n)(1-2/n)\cdots(1-(s-1)/n)]^{1/\beta}} \\ & \times \left[\frac{1}{(n-s)!}\right]^{1/\beta} \end{aligned}$$

$$\leqslant c_{15} \frac{k^{1-1/\alpha-s/\beta}}{[(1-1/k)(1-2/k)\cdots(1-(s-1)/k)]^{1/\beta}}$$

$$\times \left(\frac{1}{(k-s)!}\right)^{1/\beta}$$

$$= c_{15} k^{1-1/\alpha} \left(\frac{1}{k!}\right)^{1/\beta}. \tag{3.24}$$

Then $w_\alpha f \in L^r(\mathbb{R})$ and

$$\varepsilon_k(r, \alpha, f) \leqslant c_{15} k^{1-1/\alpha} \left(\frac{1}{k!}\right)^{1/\beta}$$

for large k. Hence,

$$\mu(r, \alpha, f) \leqslant \beta$$

and since $\beta > \mu(p, \alpha, f)$ was arbitrary,

$$\mu(r, \alpha, f) \leqslant \mu(p, \alpha, f). \tag{3.25}$$

Now, $p, r \geqslant 1$ were arbitrary. So, (3.25) implies that the quantity $\mu(p, \alpha, f)$ is really independent of p. This completes the proof of Theorem 1. ∎

For the proof of Theorem 2.2, observe that $f(z) = \sum a_n z^n$ is an entire function of finite order $\lambda > 0$ and type τ if and only if

$$(\tau\lambda)^{1/\lambda} = \limsup_{n\to\infty} \{n!^{1/\lambda} |a_n|\}^{1/n} \tag{3.26}$$

[2, Theorem 2.2.10].

The proof then proceeds on exactly the same lines as that of Theorem 2.1 and hence, the details are omitted.

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